STRAINS

- Axial Strains ✓
- Shear Strains
- Lateral Strains.

Lateral Strains

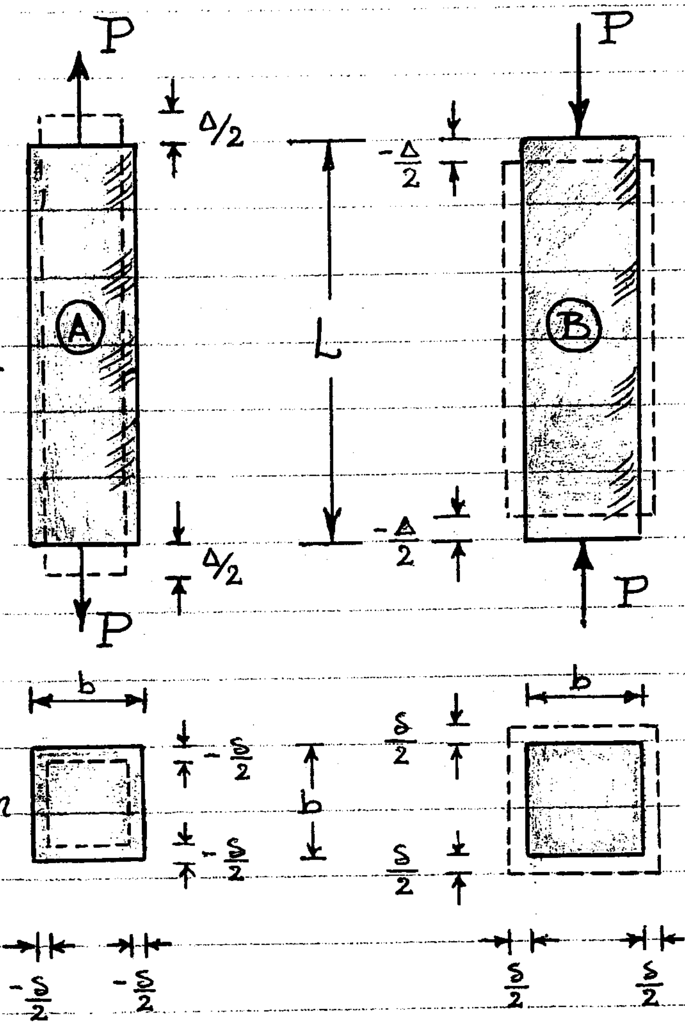
Tensile Force (P)

Axial Elongation

$\varepsilon_x = \frac{\Delta}{L}$

Lateral Contraction

$\varepsilon_{lateral} = \frac{-\delta}{b}$



Compressive Force (P)

Axial Shortening

$\varepsilon_x = \frac{-\Delta}{L}$

Lateral Expansion

$\varepsilon_{lateral} = \frac{\delta}{b}$

For Elastic, Homogeneous, and Isotropic material, define a definite property of material, just like "E", called Poisson's Ratio "ν".

$$\nu = \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = - \frac{\text{lateral strain}}{\text{axial strain}}$$

$$\Rightarrow \boxed{\nu = -\frac{\epsilon_{lateral}}{\epsilon_x} \Rightarrow \epsilon_{lateral} = -\nu\, \epsilon_x .}$$

- $\nu = 0.1 \longrightarrow$ Concrete
- $= 0.2 \rightarrow 0.3 \longrightarrow$ Steel
- $= 0.5 \longrightarrow$ Rubber

- Theoritical values of ν varies between 0 and 0.5.

$$\nu = 0 \implies \epsilon_{lateral} = 0 \quad \text{... (no lateral strains).}$$

$$\nu = 0.5 \implies \epsilon_{lateral} = -0.5\,\epsilon_x .$$

Consider Bar (A).

$$\text{Initial Volume} = V_i = b^2 * L .$$

$$\text{Final Volume} = V_f = (b-\delta)^2 * (L+\Delta).$$

$$\Rightarrow V_f = (b^2 + \delta^2 - 2b\delta) * (L+\Delta) = b^2L + b^2\Delta + \underbrace{\cancel{\delta^2 L}}_{\simeq 0} + \underbrace{\cancel{\delta^2\Delta}}_{\simeq 0} - 2b.\delta.L - \underbrace{\cancel{2b.\delta.\Delta}}_{\simeq 0}.$$

$$\Rightarrow V_f \simeq b^2L + b^2\Delta - 2b.\delta.L .$$

Dilatation $= \dfrac{\Delta V}{V} =$ Change in Volume per unit volum $= \dfrac{\Delta V}{V}$.

$$\Rightarrow \Delta V = V_f - V_i = b^2\Delta - 2b.\delta.L .$$

But $\epsilon_{lateral} = -\nu\,\epsilon_x \implies -\dfrac{\delta}{b} = -\nu\dfrac{\Delta}{L} \implies \delta = \dfrac{\nu.b.\Delta}{L}$.

$$\Rightarrow \Delta V = b^2\Delta - 2\nu.b^2\Delta = b^2\Delta.(1-2\nu).$$

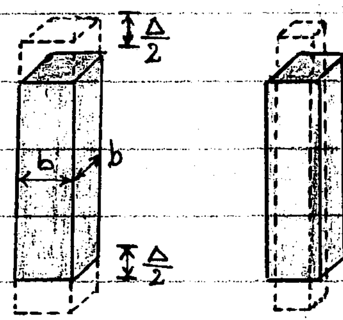
cross-sectional Area

$$\Rightarrow \boxed{\Delta V = b^2\Delta\,(1-2\nu)}$$

again; $\nu = 0 \implies \Delta V = b^2\Delta$

$\nu = 0.5 \implies \Delta V = 0.$

(Incompressible).



$\nu = 0$ $\qquad$ $\nu > 0$

Volumetric Strain

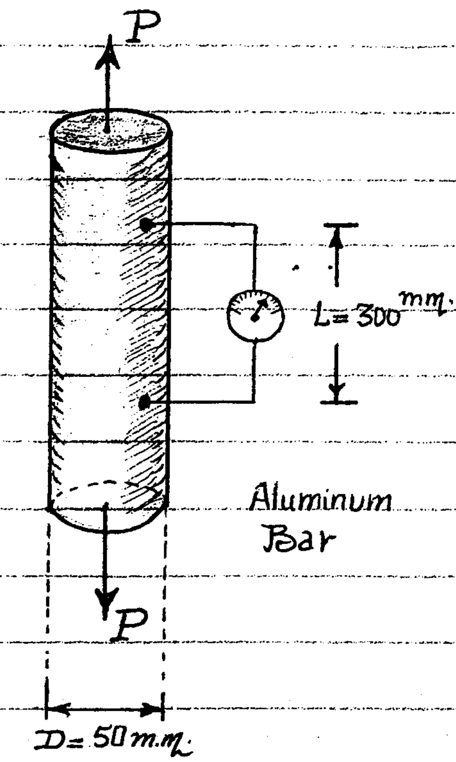
$$e_{Vol.} = \frac{\Delta V}{V} = \frac{\cancel{b^2}\Delta(1-2\nu)}{\cancel{b^2}.L}$$

$$= \frac{\Delta}{L} * (1-2\nu) \implies \boxed{e_{Vol.} = \epsilon_{axial} * (1-2\nu)}$$

Recall, $\quad \Delta = \dfrac{PL}{EA}$ and $\epsilon_x = \dfrac{\Delta}{L} = \dfrac{P}{EA} = \dfrac{\sigma_x}{E}$.

$$\Rightarrow \quad \epsilon_{lateral} = -\nu\,\epsilon_x = -\nu\frac{P}{EA} = -\nu\frac{\sigma_x}{E}.$$

Example



Experimental Data : $P = 100\,kN$

axial elong. $\Rightarrow$ $\Delta = 0.219$ m.m.

lateral reduction $\Rightarrow$ $\delta D = D_f - D_i = -0.01215$ m.m.

compute ν and E ? and ΔV ?

$$\epsilon_x = \frac{\Delta}{L} = \frac{0.219^{mm}}{300^{mm}} = 0.00073\ mm/mm$$

$$\epsilon_{lateral} = -\frac{0.01215^{mm}}{50^{mm}} = -0.000243\ mm/mm$$

- $$\nu = -\frac{\epsilon_{lateral}}{\epsilon_x} = -\frac{-0.000243}{0.00073} = \underline{\underline{0.333}}$$

- $E = ?$ $$E = \frac{\sigma_x}{\epsilon_x} = \frac{P}{\epsilon_x . A} = \frac{100^{kN}}{0.00073 * \frac{\pi (50)^2}{4} mm^2} \approx 70\ kN/mm^2 = \underline{\underline{70\ GPa}}$$

- Dilatation, $\frac{\Delta V}{V} = ?$

$\Delta V = V_f - V_i$

$V_i = \frac{\pi D_i^2}{4} * L_i = \frac{\pi (50)^2}{4} * 300 = 589048.6\ mm^3$

$V_f = \frac{\pi D_f^2}{4} * L_f$

$D_f = 50 - 0.01215 = 49.98785\ mm$

$\left.\begin{array}{l} A_f = \frac{\pi D_f^2}{4} = 1962.541\ mm^2 \\ L_f = 300 + 0.219 = 300.219\ mm \end{array}\right\} \Rightarrow V_f = 589192.2\ mm^3$

$\Rightarrow \Delta V = 589192.2 - 589048.6 = \underline{\underline{143.6\ mm^3}} \Rightarrow \frac{\Delta V}{V} = 0.0002438\ \frac{mm^3}{mm^3}$

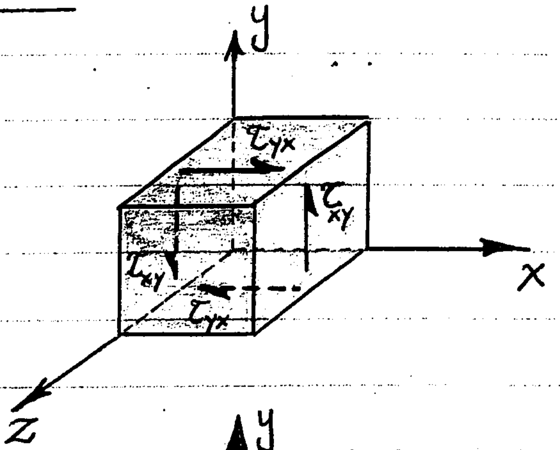
OR simply ; $\Delta V = Area * \Delta * (1 - 2\nu)$

$= \frac{\pi (50)^2}{4} * 0.219 * (1 - 2 * 0.333) = \underline{\underline{143.3\ m.m}}$

& $e = \epsilon_{axial} * (1 - 2\nu) = 0.00073 * (1 - 2(0.333)) = 0.0002433\ \frac{mm^3}{mm^3}$

Shear Strain

Stress - Strain Relationship for Shear

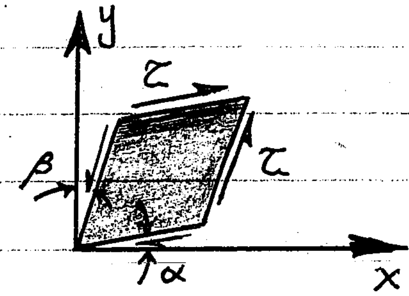


Pure shear - Planar case (2D).

↓ No elongation

↓ $\tau_{xy} = \tau_{yx} = \tau$.

Shear stress (τ) ⟶ Shear Strain (γ)

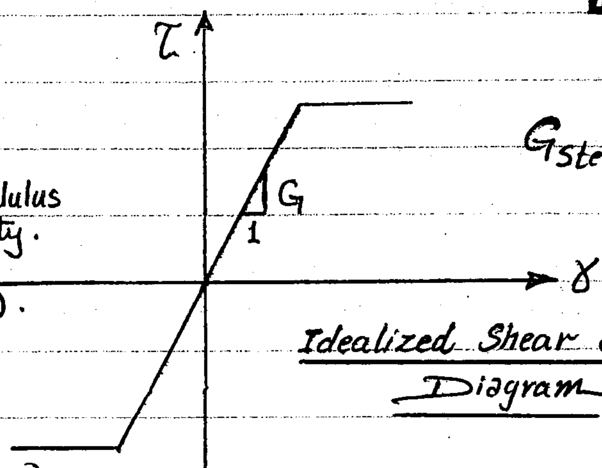


shear strain measures distortion in angle.

shear strain, $\gamma_{xy} = \alpha + \beta$.

$\tau_{xy} = G . \gamma_{xy}$

G ← Shear Modulus of elasticity. (Modulus of Rigidity).



$G_{steel} = 83\ GPa$.

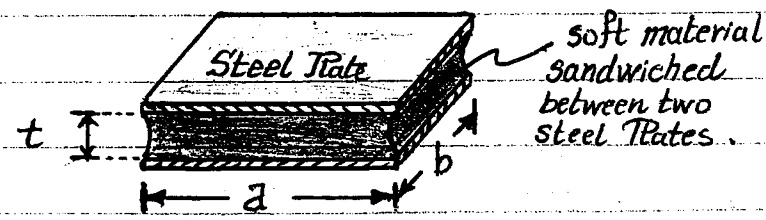
Idealized Shear Stress - Strain Diagram

$\boxed{\tau = G . \gamma}$

Extension of Hooke's Law.

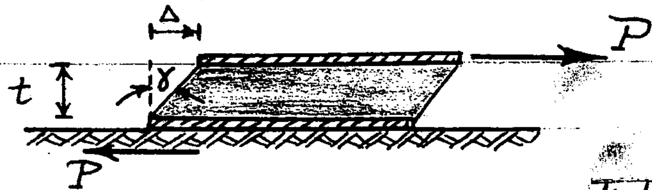
($\sigma_x = E . \epsilon_x$).

❀ Example - Pure Shear

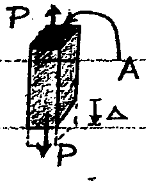
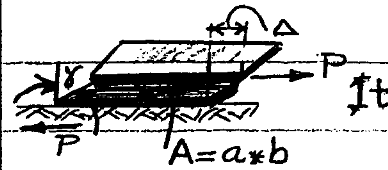


$\tau = G . \gamma$

$\tau = \dfrac{P}{a . b}$ and $\gamma = \dfrac{\Delta}{t}$

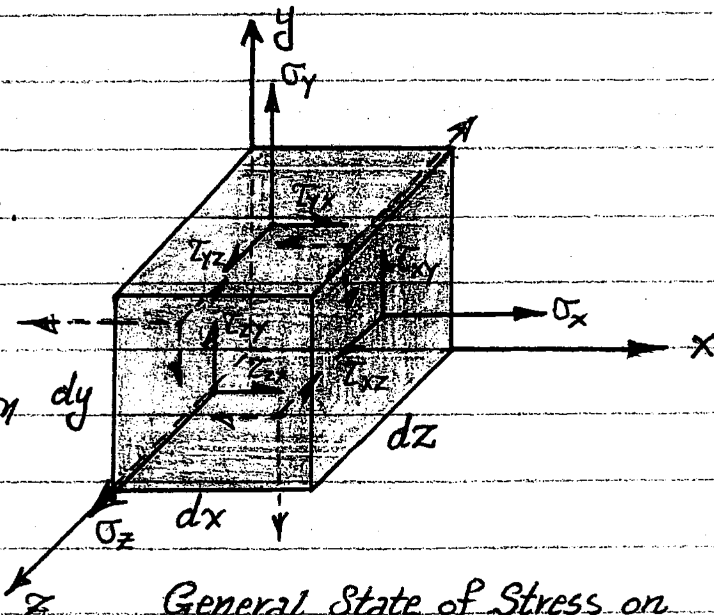


$\Rightarrow G = \dfrac{\tau}{\gamma} = \dfrac{P/a.b}{\Delta/t} = \dfrac{Pt}{\Delta . (ab)}$ ⟶ Recall, $E = \dfrac{PL}{\Delta . A}$

	Comparison between Axial & Shear (σ vs τ)	
Function	Pure Axial 	Pure Shear 
Stress	$\sigma_x = \dfrac{P}{A_{cross\text{-}section}}$	$\tau = \dfrac{P}{A_{surface}}$
Strain	$\epsilon_x = \dfrac{P}{E.A}$	$\gamma = \dfrac{P}{G.A}$
Hook's Law	$E = \dfrac{\sigma_x}{\epsilon_x}$	$G = \dfrac{\tau}{\gamma}$
Deformation (Δ)	$\Delta = \dfrac{PL}{EA}$	$\Delta = \dfrac{Pt}{G.A}$

Generalized Hooke's Law For Isotropic Materials

Consider an infinitesmal element; dx, dy, dz as shown. The general state of stress and strain are synthesized using the principle of superposition from stress-strain relationships established earlier. A set of Six equations, referred to as Generalized Hooke's Law.



General State of Stress on an infinitesmal element.

These eqns. are applicable only for elastic, homogeneous and isotropic material.

Elastic Material : Material regain original shape after unloading.

Stress and strain are proportional ; $E = \frac{\sigma}{\epsilon}$.

Homogeneous Material : Material properties are the same at all points of the body.

Isotropic Material : Material having same properties in all directions at a pt.

(same stress-strain relationship in all directions at a point.)

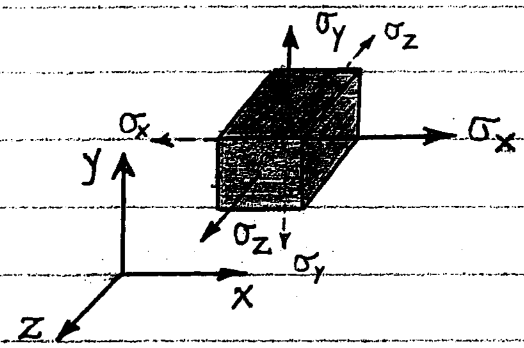
Isotropic material

$\Downarrow$

at a point

$\Downarrow$

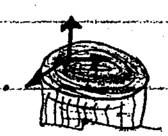
same σ-ϵ relationship in x, y, and z directions.



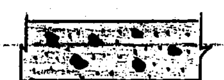
Examples

Steel $\longrightarrow$ Homogeneous and Isotropic.

Wood $\longrightarrow$ Non-homogeneous and Anisotropic.



Concrete $\longrightarrow$ Homogeneous (or Non-homogeneous), Isotropic



It should be noted that any body can be treated as a homogeneous body if the scale at which the analysis is conducted is made sufficiently large (naked eye level, grain level, atomic level, or crystal-size level).

* Due to σ_x $\longrightarrow$

$$\epsilon_x = \frac{\sigma_x}{E}$$

$$\epsilon_y = -\nu \frac{\sigma_x}{E}$$

$$\epsilon_z = -\nu \frac{\sigma_x}{E}.$$

* Due to σ_y $\longrightarrow$

$$\epsilon_x = -\nu \frac{\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E}$$

$$\epsilon_z = -\nu \frac{\sigma_y}{E}.$$

* Due to σ_z $\longrightarrow$

$$\epsilon_x = -\nu \frac{\sigma_z}{E}$$
$$\epsilon_y = -\nu \frac{\sigma_z}{E}$$
$$\epsilon_z = \frac{\sigma_z}{E}.$$

Due to triaxial loading, σ_x, σ_y, and σ_z, and by superposition:
(Linear elastic, homogeneous, and isotropic material).

Axial Strains ... [Due to σ_x, σ_y, & σ_z].

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \quad \text{------} \quad [1]$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E} \quad \text{------} \quad [2]$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \quad \text{------} \quad [3]$$

Shear strains [$\tau_{xy} = \tau_{yx}$; $\tau_{xz} = \tau_{zx}$; $\tau_{yz} = \tau_{zy}$].

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \text{------} \quad [4]$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} \quad \text{------} \quad [5]$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \quad \text{------} \quad [6]$$

❀ E, G, and ν are physical properties of the Material.

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

Dilatation and Bulk Modulus.

The sides dx, dy, and dz of the infinitesimal element considered before become $(1+\epsilon_x)dx$, $(1+\epsilon_y)dy$, and $(1+\epsilon_z)dz$, respectively, after straining.

$$\Rightarrow V_f = dx.dy.dz.(1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z).$$

$$= dx.dy.dz.[1+\epsilon_y+\epsilon_x+\cancel{\epsilon_x\epsilon_y}^{\simeq 0}+\epsilon_z+\cancel{\epsilon_z\epsilon_y}^{\simeq 0}+\cancel{\epsilon_z\epsilon_x}^{\simeq 0}+\cancel{\epsilon_x.\epsilon_y.\epsilon_z}^{\simeq 0}].$$

$$\simeq dx.dy.dz\ [1+\epsilon_x+\epsilon_y+\epsilon_z].$$

$V_i = dx.dy.dz$

$$\Rightarrow \Delta V = V_f - V_i = \underbrace{(\epsilon_x+\epsilon_y+\epsilon_z)}_{e} dx.dy.dz$$

$$\boxed{e = \text{Volumetric strain} = \text{Dilatation} = \frac{\Delta V}{V_i} = \epsilon_x+\epsilon_y+\epsilon_z}$$

Note: Shear strains cause no volume change.

By adding Eqns. [1], [2], and [3].

$$\Rightarrow e = \epsilon_x+\epsilon_y+\epsilon_z = \left(\frac{1-2\nu}{E}\right)[\sigma_x+\sigma_y+\sigma_z].$$

For an elastic body under uniform intensity of stresses, $\sigma_x=\sigma_y=\sigma_z=-p$ such as bodies subjected to hydrostatic pressure.

$$\Rightarrow e = -\frac{3(1-2\nu)}{E} * p.$$

$\bar{\sigma} = k * e$ ($\bar{\sigma}$ = average stress) (similar to $\sigma = E*\epsilon$)

OR $$\boxed{\frac{-p}{e} = k = \frac{E}{3(1-2\nu)}}$$

where k = modulus of compression (Bulk Modulus).

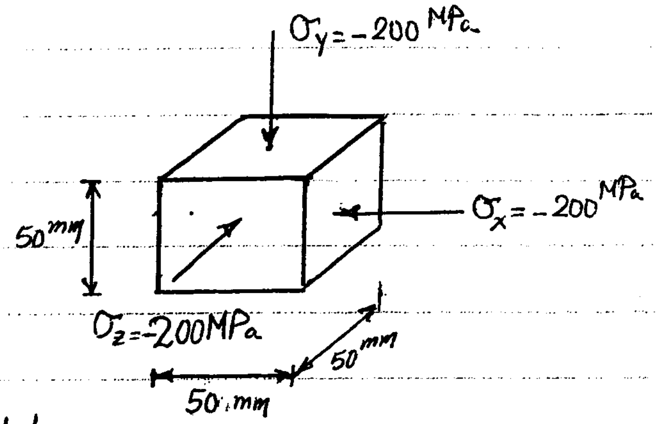
& p = average stress $= \frac{1}{3}(\sigma_x+\sigma_y+\sigma_z)$

note: $\nu = 0 \Rightarrow k = \frac{E}{3}$

$\nu = 0.5 \Rightarrow k \rightarrow \infty$ (Perfectly incompressible)

Example

A 50mm cube made of steel, $E = 200 GPa$, $\nu = 0.25$ is subject hydrostatic pressure of uniform intensity $p = -200$ MPa. Determine change in dimensions of the cube, and compute its bulk modulus.



$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} = \frac{-200}{E}(1-2\nu) = \frac{-200^{MPa}}{200*10^3}(1-2*0.25)$$
$$= -0.0005 \ mm/mm = -0.05\%$$

$$\Delta_x = \Delta_y = \Delta_z = \epsilon_x * L = -0.0005\frac{mm}{mm} * 50\,mm = \underline{\underline{-0.025\,mm}}.$$

← Reduction in dimension

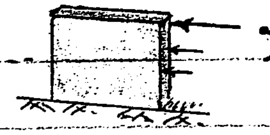
$$\bar{\sigma} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = -200 \text{ MPa}.$$
$$e = \epsilon_x + \epsilon_y + \epsilon_z = 3\epsilon_x = -3(0.05\%) = -0.15\% = -0.0015.$$
p or $\bar{\sigma}$ $\Rightarrow$ $\bar{\sigma} = k * e$.
$$\Rightarrow k = \frac{\bar{\sigma}}{e} = \frac{-200 \text{ MPa}}{-0.0015} = \underline{\underline{133333.33 \text{ MPa}}} = 133.33 \text{ GPa}$$

OR simply, $k = \frac{E}{3(1-2\nu)} = \frac{200 \text{ GPa}}{3(1-2(0.25))} = 133.33$ GPa. Ok.

Special Cases

(1) **Plane Stress** = All stresses (axial & shear) related to z-axis are equal to zero.
$$\Rightarrow \sigma_z = \tau_{xz} = \tau_{yz} = 0.$$

(2) **Plane Strain** = All strains (axial & shear) related to z-axis are equal to zero.
$$\Rightarrow \epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

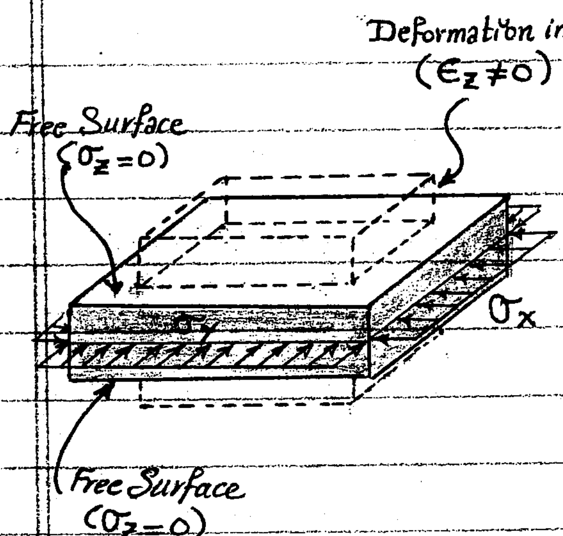
[1] Plane Stress (Ex. Structural Shear Walls )

τ_{xx}

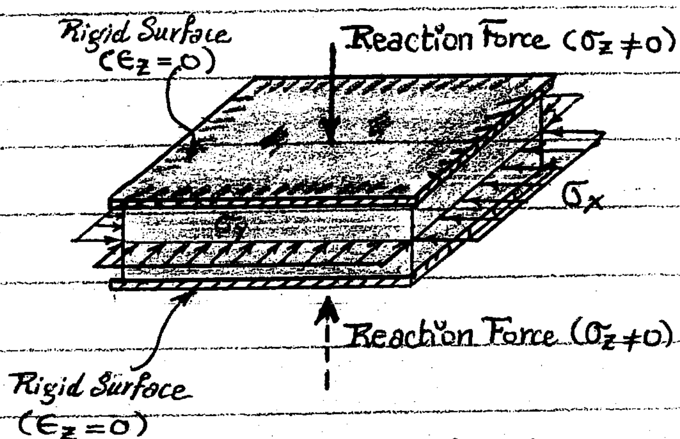
$$\begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow[\text{Hooke's Law}]{\text{Generalized}} \begin{bmatrix} \epsilon_x & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

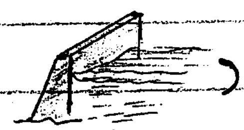
(From Eqn [3])



Plane Stress



Plane Strain

[2] Plane Strain (Ex. Gravity Earth Dams  & Pressure Vessels )

$$\begin{bmatrix} \epsilon_x & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow[\text{Hooke's Law}]{\text{Generalized}} \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

(From Eqn [3])

Note

It should be emphasized that Plane Stress & Plane Strain are often approximations utilized to simplify analysis procedure.